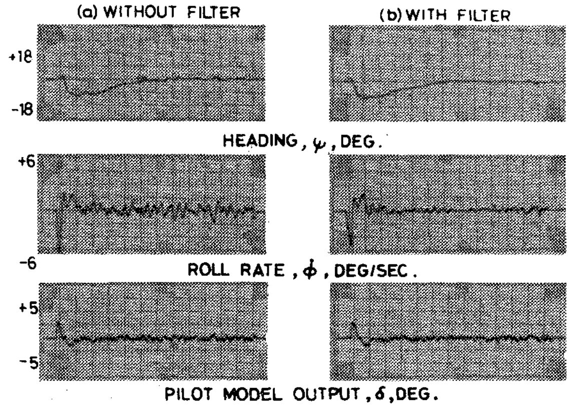

Fig. 2 Pilot model/aircraft simulated response with $\omega_c = 10$ rad/s, X axis = 1 mm/s.

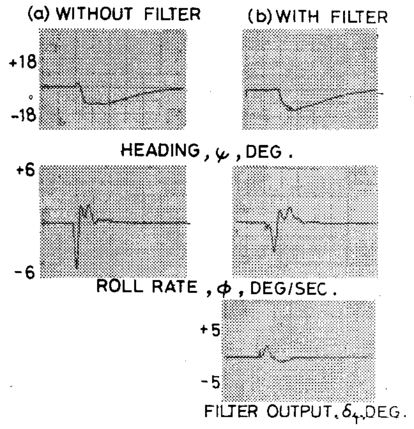

Fig. 3 Pilot model/aircraft simulated response with $\omega_c = 7.5$ rad/s, X axis = 1 mm/s.

ever be required. This point should be as low as possible for better resolution.

$$V_b = K_2 V_a^2 \qquad V_c = K(1 - V_b^2)$$

$$V_d = 0 \qquad \text{for} \quad \omega_i \le \omega_c$$

$$= V_c - V_B \qquad \text{for} \quad \omega_i > \omega_c$$

$$V_e = V_L = V_{\text{REF}} \text{ (i.e, coefficient } q = 1) \qquad \text{for} \quad \omega_i \le \omega_c$$

$$= V_L - K\left(1 - \frac{K_1^2 K_2}{\omega_i^2}\right) + V_B \qquad \text{for} \quad \omega_i > \omega_c$$

where K_1 and K_2 are gains associated with first and second stages in Fig. 1d. Thus, the value of q will be unity until the cutoff frequency and roll-off rate is 40 dB/dec. Because of uncertainties in K_1 and K_2 associated with the analog circuitry, with $K = 10$, the break point was experimentally adjusted until the desired frequency response was obtained. It may be noted that the lowest value that V_e may reach is V_B, but even this was shifted to zero at a subsequent stage. From Fig. 1a it may be seen that the phase shift introduced by the filter will always be zero.

Simulated Test

To test the proposed filter, results of pilot models obtained through model matching in a pilot/aircraft system study by Sarma and Adams[2] have been utilized. A five degree-of-freedom, nonlinear aircraft model was used on the analog computer to simulate a typical high-wing single-engine aircraft. Typical simulated test results with and without the filter in a situation where the pilot is applying a step displacement localizer correction in a path following task using an instrument landing system are shown in Fig. 2. The magnitude of the remnant added and the pilot model parameters correspond to the fixed-base simulator test result.

It can be seen from Fig. 2b that the addition of the filter at the pilot model output (in the bank angle loop) reduces the roll rate and produces a smoother response. The filter cutoff frequency in this case was set at 10 rad/s, which will have 40 dB/dec roll-off rate. Figure 3 shows another case where no remnant is added, but the filter has a reduced bandwidth. Test result shows considerable reduction in roll rate $\dot{\phi}$ without any effect on the pilot/aircraft system stability. However, as the bandwidth is reduced, small glitches were observed in the filter output and roll rate responses, which are not reflected in the heading ψ and displacements due to integration. It may be possible to avoid these glitches with further study. Tests were also conducted to study the frequency response with complex high frequencies superimposed on a low-frequency sinusoid and ramp response tests with satisfactory results.

Conclusions

The configuration of a particular nonlinear filter which does not introduce any phase shift and has good attenuation characteristics has been studied for possible applications in fly-by-wire pilot/aircraft systems. In principle, the bandwidth can be made adaptive to suit a particular flight phase. A roll-off rate of 40 dB/dec can be easily obtained. In practice, it is not difficult to obtain an accurate stick rate signal using an appropriate transducer, along with stick displacements. Hence the implementation of the filter should prove to be easy enough. Simulated tests show encouraging results and full-scale tests on a fixed-base simulator with the participation of qualified pilots is warranted.

References

[1] Adams, J.J., "Study of the Use of a Nonlinear Rate Limited Filter on Pilot-Control Signals," NASA TP-1147, April 1978.

[2] Sarma, G.R. and Adams, J.J., "Simulator Evaluation of Separation of Display Parameters in Path-Following Tasks," NASA TP-1915, Oct. 1981.

Gravitational Three-Body Problem in 120 deg Axial Coordinates

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Introduction

RECENTLY the author introduced[1] a symmetrical system of distance coordinates for studying the planar motion of a three-body system of point masses in its internal degrees of freedom. This work, devoted largely to the small vibration problem, is expanded here to give the details of a complete numerical solution of the gravitation problem, including the three angular degrees of freedom. The three coordinate axes pass through the masses with the positive halves intersecting at angles of 120 deg with each other at a moving origin. The distance coordinates are taken as the signed distances of the masses from the moving origin. If the angular orientation of the coordinate axes is defined by the Euler angles, an attempt

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to determine these angles by integrating their time rates of change fails because of singularities. It is shown that such singularities are avoided by the use of Rodrigues' orthogonal matrix to specify the angular orientation. The Lagrange equations are solved for the appropriate time derivatives, separating the distance variables from themselves and the angular velocities, thus making possible a simple computer numerical solution.

Geometrical Formulation

Figure 1 shows the masses m_1, m_2, and m_3 at the vertices of a triangle with sides y_1, y_2, and y_3, which subtend angles of 120 deg at a moving origin 0. Note that 0 is *not* identified with the center of mass. The coordinates x_1, x_2, and x_3 are taken as the signed distances from 0 to the masses. Only the positive halves of the $0x_1$, $0x_2$, and $0x_3$ coordinate axes are shown in Fig. 1. The right-handed body triad of unit vectors ***ijk*** is taken with ***k*** normal to the plane of the masses, and ***i*** pointing from 0 to m_1. The angular space orientation of the ***ijk*** triad is determined by the aerodynamicist's version of the successive Euler rotations ψ, θ, and ϕ, where ψ is a right-handed rotation about ***j***, θ is a right-handed rotation about the carried position of ***i***, and ϕ is a right-handed rotation about the carried position of ***k***.

The transformation from x_1,x_2,x_3 to the intermass distances y_1,y_2,y_3 is given by the law of cosines. The inverse transformation from y_1,y_2y_3 to x_1,x_2,x_3 was found by Bleick[1] to be

$$x_1\sqrt{3} = -y_1\cos\theta_1 + y_2\cos\theta_2 + y_3\cos\theta_3 \tag{1}$$

$$x_2\sqrt{3} = y_1\cos\theta_1 - y_2\cos\theta_2 + y_3\cos\theta_3 \tag{2}$$

$$x_3\sqrt{3} = y_1\cos\theta_1 + y_2\cos\theta_2 - y_3\cos\theta_3 \tag{3}$$

where

$$\cos\theta_1 = \pm[1-(y_2^2-y_3^2)^2 f/8y_1^2]^{1/2} \tag{4}$$

$$f = \{y_1^2+y_2^2+y_3^2+[6(y_1^2y_2^2+y_2^2y_3^2+y_3^2+y_1^2) - 3(y_1^4+y_2^4+y_3^4)]^{1/2}\}/R \tag{5}$$

$$R = y_1^4+y_2^4+y_3^4-y_1^2y_2^2-y_2^2y_3^2-y_3^2y_1^2 \tag{6}$$

and similarly for $\cos\theta_2$ and $\cos\theta_3$. Each of x_1, x_2, and x_3 may range from $-\infty$ to $+\infty$. From the area of the triangle of masses it follows that the vanishing of $x_1x_2+x_2x_3+x_3x_1$ is the condition for collinearity of the masses.

To find the kinetic energy of the masses we need the relative position vectors r_{12} from m_1 to m_2, r_{23} from m_2 to m_3, and r_{31} from m_3 to m_1. These are

$$r_{12} = i(-x_1-x_2/2) + j\sqrt{3}x_2/2 \tag{7}$$

$$r_{23} = i(x_2-x_3)/2 + j\sqrt{3}(-x_2-x_3)/2 \tag{8}$$

$$r_{31} = i(x_1+x_3/2) + j\sqrt{3}x_3/2 \tag{9}$$

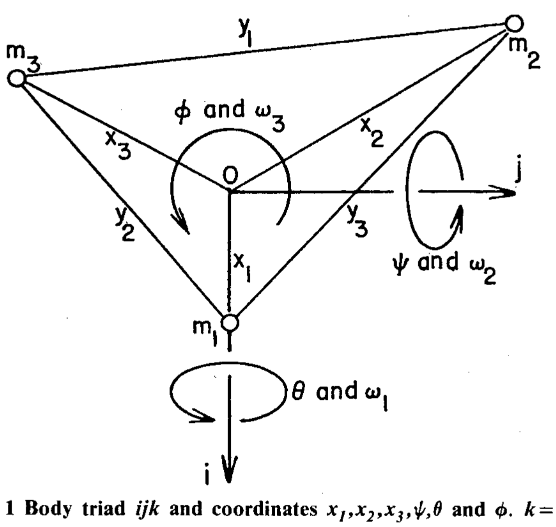

Fig. 1 Body triad *ijk* and coordinates x_1,x_2,x_3,ψ,θ and ϕ. $k=i\times j$.

According to Fraser et al.,[2] the angular velocity of the ***ijk*** triad is

$$\omega = i\omega_1 + j\omega_2 + k\omega_3 \tag{10}$$

$$\omega_1 = \dot\theta\cos\phi + \dot\psi\cos\theta\sin\phi \tag{11}$$

$$\omega_2 = -\dot\theta\sin\phi + \dot\psi\cos\theta\cos\phi \tag{12}$$

$$\omega_3 = \dot\phi - \dot\psi\sin\theta \tag{13}$$

The numbering of the axes in Eqs. (11-13) has been changed from that of Ref. 2 to be consistent with Fig. 1. If Eqs. (11-13) are solved for $\dot\psi$, $\dot\theta$, and $\dot\phi$, we find

$$\dot\theta = \omega_1\cos\phi - \omega_2\sin\phi \tag{14}$$

$$\dot\psi = (\omega_1\sin\phi + \omega_2\cos\phi)/\cos\theta \tag{15}$$

$$\dot\phi = \omega_3 + (\omega_1\sin\phi + \omega_2\cos\phi)\tan\theta \tag{16}$$

where $\dot\psi$ and $\dot\phi$ are singular at $\theta = \pm\pi/2$, making the calculation of ψ and ϕ by numerical integration impossible. It is proposed that this type of singularity be avoided by making use of the orientation coordinates afforded by the orthogonal matrix of Rodrigues,[3] more recently discussed by Turnbull.[4] This matrix may be described in terms of unit vectors by imagining that the current position of the ***ijk*** body triad has resulted from a rotation through the angle α from its initial ***i*j*k**** reference position (fixed in space) about an axis of rotation having the direction numbers *a*, *b*, and *c*. The moving vectors are then described in terms of the fixed vectors by

$$i = i^*(1+a^2-b^2-c^2)/r + j^*2(ab-c)/r + k^*2(ac+b)/r \tag{17}$$

$$j = i^*2(ab+c)/r + j^*(1-a^2+b^2-c^2)/r + k^*2(bc-a)/r \tag{18}$$

$$k = i^*2(ac-b)/r + j^*2(bc+a)/r + k^*(1-a^2-b^2+c^2)/r \tag{19}$$

where

$$r = 1+a^2+b^2+c^2 \tag{20}$$

$$|\tan(\alpha/2)| = (a^2+b^2+c^2)^{1/2} \tag{21}$$

where each of *a*, *b*, and *c* may range from $-\infty$ to $+\infty$. A way of avoiding the apparent singularity at $\alpha = \pm\pi$ is indicated later. The angular velocity in the Rodrigues coordinates may be computed to be

$$\omega = i(k\cdot dj/dt) + j(i\cdot dk/dt) + k(j\cdot di/dt) \tag{22}$$

$$\omega_1 = 2(-\dot a + c\dot b - b\dot c)/r \tag{23}$$

$$\omega_2 = 2(-\dot b + a\dot c - c\dot a)/r \tag{24}$$

$$\omega_3 = 2(-\dot c + b\dot a - a\dot b)/r \tag{25}$$

On solving Eqs. (23-25) for $\dot a$, $\dot b$, and $\dot c$, one finds

$$\dot a = -[(1+a^2)\omega_1 + (ab+c)\omega_2 + (ac-b)\omega_3]/2 \tag{26}$$

$$\dot{b} = -[(ab-c)\omega_1 + (1+b^2)\omega_2 + (bc+a)\omega_3]/2 \tag{27}$$

$$\dot{c} = -[(ac+b)\omega_1 + (bc-a)\omega_2 + (1+c^2)\omega_3]/2 \tag{28}$$

free of the singularities noted in the Euler angle coordinates. An identity connecting Eqs. (23-28) is

$$a\omega_1 + b\omega_2 + c\omega_3 = -2(a\dot{a} + b\dot{b} + c\dot{c})/r \tag{29}$$

Energy Forms

The kinetic energy T of the system may be stated in terms of the absolute time rates of change of the relative position vectors as

$$T = [m_1 m_2 (\mathrm{d}r_{12}/\mathrm{d}t)^2 + m_2 m_3 (\mathrm{d}r_{23}/\mathrm{d}t)^2 + m_3 m_1 (\mathrm{d}r_{31}/\mathrm{d}t)^2]/2m + T_c \tag{30}$$

where m is the sum of the masses and T_c is the kinetic energy of translation of the mass center. We assume that the masses move in a conservative force field so that T_c is constant and does not appear in the equations of motion. Henceforth we neglect T_c. On substituting Eqs. (7-9) in Eq. (30) we find

$$\begin{aligned} T = \{ & m_1 m_2 [\dot{x}_1^2 + \dot{x}_1\dot{x}_2 + \dot{x}_2^2 + \omega_3^2(x_1^2 + x_1x_2 + x_2^2) \\ & + \omega_3\sqrt{3}(x_2\dot{x}_1 - x_1\dot{x}_2) + O_{12}^2] + m_2 m_3 \\ & \times [\dot{x}_2^2 + \dot{x}_2\dot{x}_3 + \dot{x}_3^2 + \omega_3^2(x_2^2 + x_2x_3 + x_3^2) \\ & + \omega_3\sqrt{3}(x_3\dot{x}_2 - x_2\dot{x}_3) + O_{23}^2] + m_3 m_1 [\dot{x}_3^2 + \dot{x}_3\dot{x}_1 + \dot{x}_1^2 \\ & + \omega_3^2(x_3^2 + x_3x_1 + x_1^2) + \omega_3\sqrt{3}(x_1\dot{x}_3 - x_3\dot{x}_1) + O_{31}^2]\}/2m \end{aligned} \tag{31}$$

where

$$O_{12} = \omega \times r_{12}\cdot k, \quad O_{23} = \omega\times r_{23}\cdot k, \quad O_{31} = \omega\times r_{31}\cdot k \tag{32}$$

The gravitational potential energy is

$$V = -G(m_1 m_2/y_3 + m_2 m_3/y_1 + m_3 m_1/y_2) \tag{33}$$

where G is the gravitational constant.

Lagrange Equations

The Lagrange equations for x_1, x_2, and x_3 are

$$\begin{aligned} &(m_2+m_3)(\ddot{x}_1 - \omega_3^2 x_1) + m_2(\ddot{x}_2 - \omega_3^2 x_2)/2 + m_3(\ddot{x}_3 - \omega_3^2 x_3)/2 \\ &= -Gmm_2(x_1 + x_2/2)/y_3^3 - Gmm_3(x_1 + x_3/2)/y_2^3 \\ &+ T_1/2 - \sqrt{3}\dot{\omega}_3 g_1/2 - \sqrt{3}\omega_3\dot{g}_1 \end{aligned} \tag{34}$$

$$\begin{aligned} &m_1(\ddot{x}_1 - \omega_3^2 x_1)/2 + (m_3+m_1)(\ddot{x}_2 - \omega_3^2 x_2) + m_3(\ddot{x}_3 - \omega_3^2 x_3)/2 \\ &= -Gmm_3(x_2 + x_3/2)/y_1^3 - Gmm_1(x_2 + x_1/2)/y_3^3 \\ &+ T_2/2 - \sqrt{3}\dot{\omega}_3 g_2/2 - \sqrt{3}\omega\dot{g}_2 \end{aligned} \tag{35}$$

$$\begin{aligned} &m_1(\ddot{x}_1 - \omega_3^2 x_1)/2 + m_2(\ddot{x}_2 - \omega_3^2 x_2)/2 + (m_1+m_2)(\ddot{x}_3 - \omega_3^2 x_3) \\ &= -Gmm_1(x_3 + x_1/2)/y_2^3 - Gmm_2(x_3 + x_2/2)/y_1^3 \\ &+ T_3/2 - \sqrt{3}\dot{\omega}_3 g_3/2 - \sqrt{3}\omega_3\dot{g}_3 \end{aligned} \tag{36}$$

where

$$T_1 = (\partial/\partial x_1)(m_2 O_{12}^2 + m_3 O_{31}^2) \tag{37}$$

$$g_1 = m_2 x_2 - m_3 x_3 \tag{38}$$

with T_2 and T_3 obtained from T_1, and g_2 and g_3 obtained from g_1, by cyclic permutation of subscripts. Solve Eqs. (34-38) for $\ddot{x}_1$ to find

$$\begin{aligned} M(\ddot{x}_1 - \omega_3^2 x_1) &= -\sqrt{3}m(\dot{\omega}_3 s_1 + 2\omega_3\dot{s}_1)/4m_1 \\ &- m_2(m_1 + m_2 - m_3/2)T_2/4 + [(m_1+m_2)(m_1+m_3) \\ &- m_2m_3/4]T_1/2 - m_3(m_1 + m_3 - m_2/2)T_3/4 \\ &+ 3Gmm_2m_3[(m_1 + m_2/2)x_2 + (m_1 + m_3/2)x_3]/4y_1^3 \\ &- 3Gmm_3\{[mm_1 + m_2(m_3 + m_1/2)]x_1 \\ &+ m_2(m_1 + m_3/2)x_3\}/4y_2^3 - 3Gmm_2 \\ &\times\{[mm_1 + m_3(m_2 + m_1/2)]x_1 + m_3(m_1 + m_2/2)x_2\}/4y_3^3 \end{aligned} \tag{39}$$

where

$$\begin{aligned} s_1 = m_1[&m_1(m_2 - m_3)x_1 + m_2(2m_1 + m_3)x_2 \\ &- m_3(2m_1 + m_2)x_3] \end{aligned} \tag{40}$$

$$M = 3[(m_1+m_2)(m_2+m_3)(m_3+m_1) + m_1m_2m_3]/4 \tag{41}$$

with similar expressions for $\ddot{x}_2$ and $\ddot{x}_3$ by cyclic permutation of the subscripts of Eqs. (39) and (40), excluding the subscripts of ω_3 and $\dot{\omega}_3$.

The Lagrange equation for ϕ, which is useful in separating the second derivatives of the distance variables from the derivatives of the angular velocities, is

$$\begin{aligned} &\sqrt{3}[m_1 g_1(\ddot{x}_1 - \omega_3^2 x_1) + m_2 g_2(\ddot{x}_2 - \omega_3^2 x_2) \\ &+ m_3 g_3(\ddot{x}_3 - \omega_3^2 x_3)]/2 + (\mathrm{d}/\mathrm{d}t)J\omega_3 \\ &= (\partial/\partial\phi)(m_1m_2O_{12}^2 + m_2m_3O_{23}^2 + m_3m_1O_{31}^2)/2 \end{aligned} \tag{42}$$

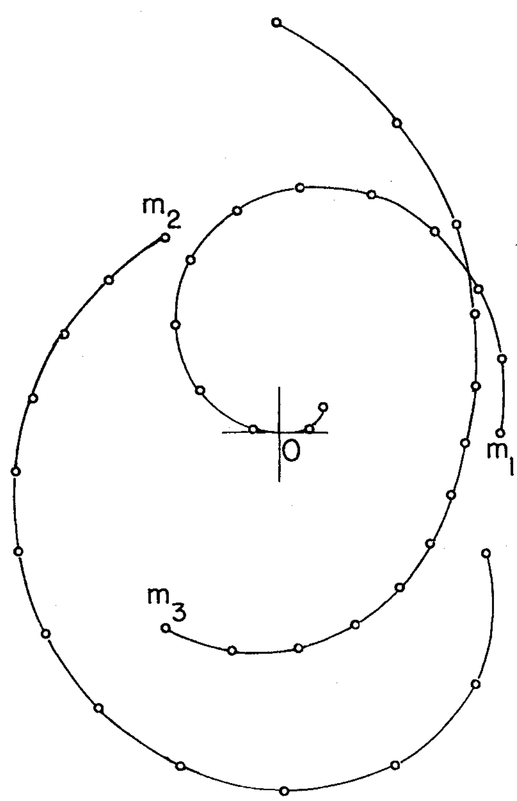

Fig. 2 Mass m_1 boosted radially out of common circle orbit with speed of 20% of its orbital velocity, $m_1 = m_2 = m_3$.

where

$$J = m_1 m_2 y_3^2 + m_2 m_3 y_1^2 + m_3 m_1 y_2^2 \tag{43}$$

Substitute the $\ddot{x}_i - \omega_3^2 x_i$ of Eqs. (39-41) into Eq. (42) to obtain the ω_3 differential equation

$$m m_1 m_2 m_3 s^2 \dot{\omega}_3 = \omega_3 (P - \gamma \dot{J}) - \sqrt{3} G Q m m_1 m_2 m_3 s/2 - U/2\sqrt{3} + \gamma (\partial / \partial \phi)(m_1 m_2 0_{12}^2 + m_2 m_3 0_{23}^2 + m_3 m_1 0_{31}^2)/2 \tag{44}$$

where

$$s = x_1 + x_2 + x_3, \qquad \gamma = 4M/3m \tag{45}$$

$$P = g_1 \dot{s}_1 + g_2 \dot{s}_2 + g_3 \dot{s}_3 \tag{46}$$

$$Q = g_1 / y_1^3 + g_2 / y_2^3 + g_3 / y_3^3 \tag{47}$$

$$U = s_1 T_1 + s_2 T_2 + s_3 T_3 \tag{48}$$

Since the scalar triple products 0_{12}, 0_{23} and 0_{31} vanish for planar motion, where $\omega_1 = \omega_2 = 0$, numerical integration of Eqs. (39) and (44) together with

$$\phi = \int \omega_3 \mathrm{d}t \tag{49}$$

gives all the information needed to plot the internal degrees-of-freedom trajectories in this case. Figure 2 illustrates the trajectories for three equal masses moving in a common circular orbit when mass m_1 is given a radial boost of 20% of its orbital velocity at zero time. The plotted circles are the positions of the masses for constant increments in time.

The relation

$$J\dot{\omega}_3 + \sqrt{3}[m_1 g_1 (\ddot{x}_1 - \omega_3^2 x_1) + m_2 g_2 (\ddot{x}_2 - \omega_3^2 x_2) + m_3 g_3 (\ddot{x}_3 - \omega_3^2 x_3)]/2 = [m m_1 m_2 m_3 s (s\dot{\omega}_3 + \sqrt{3} G Q/2) - \omega_3 P + U/2\sqrt{3}]/\gamma \tag{50}$$

may be obtained from Eqs. (42) and (44). In the following, the notation $0_{12}(\omega_1, \omega_2)$ will be used for 0_{12}, and similarly for 0_{23} and 0_{31}, whenever ω_1 and ω_2 are assigned values involving a, b, c, or a constant.

On making use of Eq. (50), the Lagrange equations for the Rodrigues orientation variables a, b, and c may be written, respectively, as

$$\begin{aligned}
& b m m_1 m_2 m_3 s^2 \dot{\omega}_3/\gamma + \sqrt{3}\dot{\omega}_1 [m_1 m_2 x_2 0_{12}(-1,-c) - m_2 m_3 (x_2 + x_3) 0_{23}(-1,-c) + m_3 m_1 x_3 0_{31}(-1,-c)]/2 \\
& + \dot{\omega}_2 [m_1 m_2 (x_1 + x_2/2) 0_{12}(-1,-c) - m_2 m_3 (x_2 - x_3) 0_{23}(-1,-c)/2 - m_3 m_1 (x_1 + x_3/2) 0_{31}(-1,-c)] \\
& = -\omega_3 [\dot{J} b + 2J\dot{b} + Jb(a\omega_1 + b\omega_2 + c\omega_3) + aJ\omega_3] - \sqrt{3} H [2\dot{b} + b(a\omega_1 + b\omega_2 + c\omega_3) + a\omega_3]/2 - m_2 [m_1 0_{12}(-1,-c) - m_3 0_{23} \\
& \times (-1,-c)][\sqrt{3}\omega_1 \dot{x}_2/2 + \omega_2(\dot{x}_1 + \dot{x}_2/2)] - m_3 [m_1 0_{31}(-1,-c) - m_2 0_{23}(-1,-c)][\sqrt{3}\omega_1 \dot{x}_3/2 - \omega_2(\dot{x}_1 + \dot{x}_3/2)] \\
& + m_2 (m_1 0_{12} - m_3 0_{23})[0_{12}(-a\omega_1, \dot{c} - a\omega_2) - (a\omega_1 + b\omega_2 + c\omega_3 + \mathrm{d}/\mathrm{d}t) 0_{12}(-1,-c)] + m_3 (m_1 0_{31} - m_2 0_{23}) \\
& \times [0_{31}(-a\omega_1, \dot{c} - a\omega_2) - (a\omega_1 + b\omega_2 + c\omega_3 + \mathrm{d}/\mathrm{d}t) 0_{31}(-1,c)] + b(\omega_3 P - U/2\sqrt{3} - \sqrt{3} G m m_1 m_2 m_3 s Q/2)/\gamma
\end{aligned} \tag{51}$$

$$\begin{aligned}
& -a m m_1 m_2 m_3 s^2 \dot{\omega}_3/\gamma + \sqrt{3}\dot{\omega}_1 [m_1 m_2 x_2 0_{12}(c,-1) - m_2 m_3 (x_2 + x_3) 0_{23}(c,-1) + m_3 m_1 x_3 0_{31}(c,-1)]/2 + \dot{\omega}_2 [m_1 m_2 (x_1 + x_2/2) 0_{12} \\
& \times (c,-1) - m_2 m_3 (x_2 - x_3) 0_{23}(c,-1)/2 - m_3 m_1 (x_1 + x_3/2) 0_{31}(c,-1)] = \omega_3 [\dot{J} a + 2J\dot{a} + Ja(a\omega_1 + b\omega_2 + c\omega_3) - bJ\omega_3] \\
& + \sqrt{3} H [2\dot{a} + a(a\omega_1 + b\omega_2 + c\omega_3) - b\omega_3]/2 - m_2 [m_1 0_{12}(c,-1) - m_3 0_{23}(c,-1)][\sqrt{3}\omega_1 \dot{x}_2/2 + \omega_2(\dot{x}_1 + \dot{x}_2/2)] \\
& - m_3 [m_1 0_{31}(c,-1) - m_2 0_{23}(c,-1)][\sqrt{3}\omega_1 \dot{x}_3/2 - \omega_2(\dot{x}_1 + \dot{x}_3/2)] + m_2 (m_1 0_{12} - m_3 0_{23})[0_{12}(-\dot{c} - b\omega_1, -b\omega_2) \\
& - (a\omega_1 + b\omega_2 + c\omega_3 + \mathrm{d}/\mathrm{d}t) 0_{12}(c,-1)] + m_3 (m_1 0_{31} - m_2 0_{23})[0_{31}(-\dot{c} - b\omega_1, -b\omega_2) - (a\omega_1 + b\omega_2 + c\omega_3 + \mathrm{d}/\mathrm{d}t) 0_{31}(c,-1)] \\
& - a(\omega_3 P - U/2\sqrt{3} - \sqrt{3} G m m_1 m_2 m_3 s Q/2)/\gamma
\end{aligned} \tag{52}$$

$$\begin{aligned}
& - m m_1 m_2 m_3 s^2 \dot{\omega}_3/\gamma + \sqrt{3}\dot{\omega}_1 [m_1 m_2 x_2 0_{12}(-b,a) - m_2 m_3 (x_2 + x_3) 0_{23}(-b,a) + m_3 m_1 x_3 0_{31}(-b,a)]/2 \\
& + \dot{\omega}_2 [m_1 m_2 (x_1 + x_2/2) 0_{12}(-b,a) - m_2 m_3 (x_2 - x_3) 0_{23}(-b,a)/2 - m_3 m_1 (x_1 + x_3/2) 0_{31}(-b,a)] \\
& = \omega_3 [\dot{J} + J(a\omega_1 + b\omega_2)] + \sqrt{3} H (a\omega_1 + b\omega_2)/2 - m_2 [m_1 0_{12}(-b,a) - m_3 0_{23}(-b,a)][\sqrt{3}\omega_1 \dot{x}_2/2 + \omega_2(\dot{x}_1 + \dot{x}_2/2)] \\
& - m_3 [m_1 0_{31}(-b,a) - m_2 0_{23}(-b,a)][\sqrt{3}\omega_1 \dot{x}_3/2 - \omega_2(\dot{x}_1 + \dot{x}_3/2)] + m_2 (m_1 0_{12} - m_3 0_{23})[0_{12}(\dot{b} - c\omega_1, -\dot{a} - c\omega_2) \\
& - (a\omega_1 + b\omega_2 + c\omega_3 + \mathrm{d}/\mathrm{d}t) 0_{12}(-b,a)] + m_3 (m_1 0_{31} - m_2 0_{23})[0_{31}(\dot{b} - c\omega_1, -\dot{a} - c\omega_2) \\
& - (a\omega_1 + b\omega_2 + c\omega_3 + \mathrm{d}/\mathrm{d}t) 0_{31}(-b,a)] - (\omega_3 P - U/2\sqrt{3} - \sqrt{3} G m m_1 m_2 m_3 s Q/2)/\gamma
\end{aligned} \tag{53}$$

where

$$H = m_1 g_1 \dot{x}_1 + m_2 g_2 \dot{x}_2 + m_3 g_3 \dot{x}_3 \tag{54}$$

Equations (51-53) may be solved numerically for $\dot{\omega}_1$, $\dot{\omega}_2$ and $\dot{\omega}_3$. Using these values, with the values of $\dot{a}$, $\dot{b}$, and $\dot{c}$ from Eqs. (26-28) and the values of $\ddot{x}_1$, $\ddot{x}_2$, and $\ddot{x}_3$ from Eqs. (39-41), the numerical integration of the internal degrees-of-freedom trajectories may now be undertaken. Use the initial values $a=b=c=0$ at zero time. The value of $|\tan\alpha/2|$ of Eq. (21) must be checked at each step of the integration to watch the approach to the singularity at $\alpha=\pm\pi$. If $|\tan\alpha/2|$ exceeds some threshold, say $\sqrt{3}$, take the current ***ijk*** body triad as a new reference frame with $a=b=c=0$, and continue the integration with no other change in the dependent variables. It will be necessary to use matrix multiplication to compute and store the direction cosines of the ***ijk*** body triad relative to the triad at zero time at each such change in reference frame.

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